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The Hagedorn spectrum and large N_c QCD in 2+1 and 3+1 dimensions

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Abstract: We show that a Hagedorn spectrum (i.e., spectrum where the number of hadrons grows exponentially with the mass) emerges automatically in large N_c QCD in 2+1 and 3+1 dimensions. The approach is based on the study of Euclidean space correlation functions for composite operators constructed from quark and gluon fields and exploits the fact that the short time behavior of the correlators is known in QCD. The demonstration relies on one critical assumption: that perturbation theory accurately describes the trace of the logarithm of a matrix of point-to-point correlation functions in the regime where the perturbative corrections to the asymptotically free value are small.

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Contents

1 Introduction

One of the oldest questions in strong interaction physics is the density of hadrons in the spectrum as a function of mass for large mass. It was conjectured long ago by Hagedorn that this density (when suitably averaged) grew exponentially with the mass [1, 2]. A useful way to parameterize this is via its integral, $N(m)$, the number of hadrons with mass less than m. One way to state Hagedorn's conjecture is that at asymptotically large m,

$$N(m) \sim \left(\frac{m}{T_H}\right)^a \exp\left(\frac{m}{T_H}\right) , \tag{1.1}$$

where T_H, the so-called Hagedorn temperature, is a parameter controlling the exponential growth. Note that in a simple model, where hadrons are treated as a noninteracting free gas, the Hagedorn temperature represents an upper bound on the temperature of a hadronic phase of matter as the energy density diverges for $T > T_H$. The power-law prefactor plays an important role in attempts to fit the Hagedorn spectrum from data [3] and also determines the thermodynamic behavior of strongly interacting matter as T_H is approached from below in the simple noninteracting hadron gas model [4]. A more useful way to state Hagedorn's conjecture for the purposes of this paper is that for asymptotically large masses there exists positive value of T such that

$$N(m) \geq e^{m/T} ; \tag{1.2}$$

T_H is the maximum value of T for which eq. (1.2) holds.

As shown in figure 1, the extracted masses of hadronic resonances [5], $N(m)$ does, indeed, grow very rapidly up to the point where it becomes difficult to extract resonance parameters from experimental data (around 2 GeV). This behavior appears to be consistent

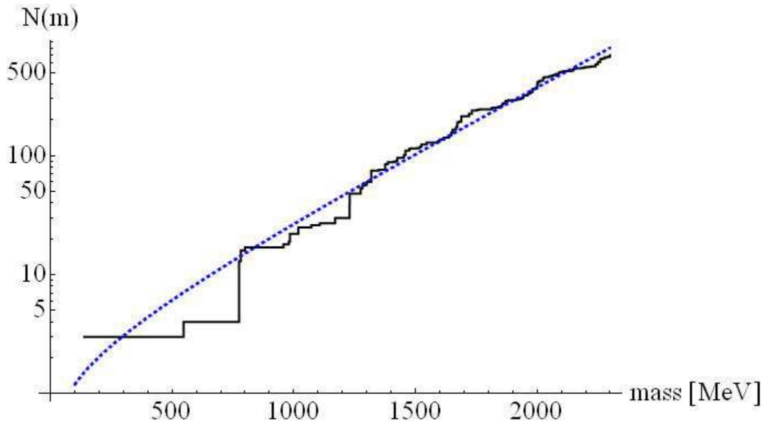

Figure 1. $N(m)$ for nonstrange mesons using mesons masses extracted from various hadronic processes extracted reported by the Particle Data Group [5]. The fit is of the form $N(m) = am^b e^{m/T_H}$ and yields a Hagedorn temperature of 426 MeV. The fits were done for mesons with masses up to 2300 MeV. Our estimate of the Hagedorn temperature is consistent with the results of ref. [3]. Note that it is almost 2.5 times larger than the critical temperature T_C obtained by lattice gauge calculations [6].

with the notion that QCD does have a Hagedorn spectrum. However, it is very difficult to establish Hagedorn's conjecture in a compelling way from the empirical data. In part this is a practical issue; one would need to extract hadron masses for hadrons up to much larger masses to get compelling evidence for an exponential growth. Moreover, underlying this practical issue is an important theoretical one: highly excited hadrons are not particles; they are resonances and as such do not have well-defined masses. The mass parameters can only be extracted from partial wave analysis of various scattering processes using some model dependent assumptions. Such model-dependence is quite weak for well-isolated narrow resonances and for these one can state masses with some level of confidence. However as resonances in some channel become wide or close to each other, such model dependence grows and it becomes difficult to isolate resonant state in a meaningful way. Moreover, *any* model dependence in the meaning of a hadron's mass makes the issue of the density of hadronic states intellectually problematic.

Before attempting to deal with the problem of ill-defined hadron masses, it is useful to understand why one might expect QCD to have a Hagedorn spectrum. Recall that Hagedorn spectra arises automatically in simple string theories [7] with unbreakable and noninteracting strings. It is noteworthy that string theory was originally formulated as a theory of strong interaction. Moreover, given confinement it is plausible that highly excited states in QCD should act stringy. For the case of pure gauge theory there is strong evidence [8] that widely separated static quark sources have a linearly rising potential, (i.e., an area law for the Wilson loop). This arises because for widely separated sources, the flux arranges itself into tubes with a characteristic width and fixed energy per unit length [8]. It is plausible that for highly excited states, which would be expected to have flux tubes which are much longer than their width would act dynamically as strings and as an

effective string theory would naturally give rise to a Hagedorn spectrum. Mesons in such a picture are interpreted as open strings. However, this picture is flawed. It is based on pure gauge theory, in which confinement implies unbreakable strings. In QCD, with dynamical quarks, flux tubes can break. Indeed, this is the same issue as noted above — the fact that flux tubes break implies that mesons decay and thus can only be seen as resonances with nonzero widths.

It is not clear whether there is a clean way to deal with this issue in an unambiguous way for QCD in the physical world. However, if one focuses on the large N_c limit of QCD [9, 10], the issue vanishes. As the large N_c limit is approached, meson decays are suppressed by a factor of $1/N_c$; flux tubes do not break and mesons become stable. The goal of the present work is to show that at large N_c QCD must have a phase transition. This is at least a well-posed theoretical question. Of course, the question of whether or not this is of phenomenological relevance depends on how close the $N_c = 3$ world is to the large N_c world.

The first derivation of a Hagedorn spectrum in some variant of large N_c QCD was done by Kogan and Zhitnitsky [11], who explicitly computed the spectrum of large N_c QCD in 1+1 dimensions with adjoint fermions and showed that it possess a Hagedorn-type behavior. Ideally one could similarly compute the spectrum for large N_c QCD for more than one spatial dimension. However, in practice we do not know how to solve for the spectrum. Numerical studies using lattice QCD are poorly suited for extracting high-lying stars. There was a study of the large N_c glueball spectrum based on a numerical treatment of a transverse lattice QCD in a light cone formalism [12]; while the results are consistent with the Hagedorn spectrum, the evidence was not definitive. There are, however, indirect ways to probe the issue. One is by the study of QCD thermodynamics. It is well known that large N_c QCD has a first order phase transition to a quark-gluon plasma phase [13, 14] with the latent heat growing as N_c^2. This transition tells us nothing about a Hagedorn spectrum. However, systems with first order transitions can superheat and thus a hadronic phase can exist about T_c. The Hagedron spectrum and the noninteracting nature of hadrons at large N_c implies a maximum temperature for this superheated hadronic phase [4, 15]. Moreover, it is practical to study this superheated phase in lattice QCD for moderately large N_c and thus get indirect evidence for a Hagedorn spectrum.

An alternative indirect way to demonstrate a Hagedorn spectrum for large N_c QCD for 3+1 dimensions was outlined in ref. [16]. The argument relies only standard and generally accepted properties of QCD. Confinement in its basic sense that all physical states are color singlets plays a critical role as does asymptotic freedom. In addition, the approach requires some plausible assumptions about the validity of perturbation theory to describe the correlation functions at short times. However, the approach explicitly assumes neither that the hadron dynamics is stringy in nature nor that the confinement is manifest through an unbroken center symmetry. The argument relies critically on the fact that the number of independent local operators with given set of quantum numbers grows exponentially with the mass dimension of operators. This approach is similar in spirit to the ideas of Kogan and Zhitnitski [11]; it also has elements which are reminiscent of refs. [17–19].

The principal purpose of this paper is to generalize the argument of ref. [16]. The

version of the argument in ref. [16] does not apply to 2+1 dimensions; here we develop a variant of the argument which is applicable to both 3+1 and 2+1 cases. We also simplify and clarify the arguments in ref. [16] and improve it in significant ways. The principal improvement is in the treatment of perturbative corrections to correlations in section 7 of this work. In ref. [16] corrections due to certain classes of diagram were shown to have a required behavior and it was suggested that the general case ought to behave similarly. Here a complete demonstration that this is true is given.

2 Outline of procedure

We start by noting that at large N_c meson widths go to zero. Thus the spectrum of mesons is unambiguously defined. To begin the analysis we define two functions characterizing the spectrum of hadrons. $N(m)$ is defined as the number of hadrons with mass less than m, and $W(m)$ defined as the sum of the masses of respective particles

$$W(m) = \sum_{i}^{N_m} m_i = mN(m) - \int_0^m \mathrm{d}\mu N(\mu) \ . \tag{2.1}$$

It is easy to see that if one of them grows exponentially so does the other.

Next, we explicitly construct a sequence of sets of local operators with fixed mass dimension (labeled with n) which grows exponentially. The number of operators in each set of the sequence is

$$N = A^n. \tag{2.2}$$

Our goal will be to demonstrate that at sufficiently large n the following inequalities hold

$$N(an + b) \geq V \ ; \tag{2.3}$$

$$V \geq W(m_N) \ , \tag{2.4}$$

where V is the negative logarithmic derivative of the trace of a certain matrix of correlators and a and b are constants with dimensions of mass. The key feature is that the left-hand side grows linearly with n and no faster.

If we now assume that the number of hadrons is bounded from above $N(m) \leq \exp(\alpha m)$ we can easily derive from eqs. (2.1)-(2.4) following expression:

$$a\alpha \log_A(e) m + b \geq m - \frac{1}{\alpha} \ . \tag{2.5}$$

As $m \to \infty$ there is a contradiction unless $\alpha \geq \frac{1}{a \log_A(e)}$ and the assumption that function $N(m)$ is bound by exponential is false. Consistency requires

$$N(m) \geq \exp\left(\frac{1}{a \log_A(e)} m\right) \ . \tag{2.6}$$

Consequently we obtain the Hagedorn spectrum.

Inequalities (2.3) and (2.4) are somewhat subtle and their derivation will be described in detail in the following sections.

3 Sets of local operators

To proceed further we need to construct sets of composite single-color-trace color-singlet local operators. The matrix of correlators of operators in these sets is the core of the argument. The single color trace nature of these operators will ensure that at large N_c each of these operators when acting on the vacuum makes a single hadron (provided that one assumes confinement) [9]. These operators need to have the property that at large N_c, the correlator between two distinct operators in the set must vanish as the distance between the operators goes to zero. For simplicity we consider operators which transform as Lorentz scalars; these are guaranteed to produce spinless hadrons when acting on the vacuum (for any spatial dimension). It is sufficient to show that that the number of spinless hadrons grows exponentially to establish a Hagedorn spectrum.

The operators we use in our construction need to be different for 2+1 and 3+1 dimensional QCD. We will construct our operators out of some basic building blocks. In 3+1 dimensions, these building blocks are the following two types of operators:

$$O_1 = \text{const} \cdot F_{\mu\nu} F^{\mu\nu} \ , \quad O_2 = \text{const} \cdot F_{\mu\nu} \widetilde{F}^{\mu\nu} \ , \tag{3.1}$$

and in 2+1 dimensions:

$$\begin{aligned} O_1 &= \text{const} \cdot F_{\alpha\beta} F^{\alpha\beta} \, F_{\alpha'\beta'} F^{\alpha'\beta'} \, F_{\alpha''\beta''} F^{\alpha''\beta''} \ , \\ O_2 &= \text{const} \cdot \left[\epsilon^{\alpha\mu\nu} F_{\mu\nu} \, \epsilon^{\beta\mu'\nu'} F_{\mu'\nu'} \, \epsilon^{\gamma\mu''\nu''} F_{\mu''\nu''} \ \epsilon_{\alpha\beta\gamma} \right]^2 \ , \end{aligned} \tag{3.2}$$

where the constants may be chosen for convenience and do not affect any results. These operators are not traced over color. Thus, in the large N_c limit these become pure color adjoint operators. It is easy to see that such operators are linearly independent—one is a scalar and one is a pseudoscalar.

From these basic building blocks, we create the individual color-singlet operators in the following way:

$$J_{l_1, l_2, \ldots, l_n} = \bar{q} O_{l_1} O_{l_2} \ldots O_{l_n} q \ , \tag{3.3}$$

where ls are either 1 or 2 and n is the total number of the operators O inserted between quarks. We construct sets of these operators each of which has the same value of n. Thus,

$$\begin{aligned} &\mathcal{S}_1 = \{J_1, J_2\} = \{\overline{q} O_1 q, \overline{q} O_2 q\} \ , \\ &\mathcal{S}_2 = \{J_{11}, J_{12}, J_{21}, J_{22}\} = \{\overline{q} O_1 O_1 q, \overline{q} O_1 O_2 q, \cdots\} \ , \\ &\mathcal{S}_3 = \{J_{111}, J_{112}, J_{121}, J_{122}, J_{211}, J_{212}, \cdots\} \ , \\ &\cdots \ . \end{aligned} \tag{3.4}$$

Ultimately we consider these a *sequence* of sets where the n^{th} element of the sequence is $\mathcal{S}_n$. We will then focus on the behavior as n becomes large. By construction, the number of currents in the n^{th} step of the sequence equals $N = 2^n$. Additionally, all currents in a given set $\mathcal{S}_n$ have the same (naive) mass dimension, $4n+3$ for 3+1 dimensions and $12n+3$ for 2+1 dimensions, respectively. To sum up, we created a sequence of sets of operators where the number of elements in each step grows exponentially with a mass dimension.

4 Matrix of current correlators

Let us define a sequence of correlator matrices $\mathbf{\Pi}^{(n)}$ between two space-time points. Their matrix elements read

$$\Pi^{(n)}_{ab}(x-y) = \left\langle J^\dagger_a(x) J_b(y) \right\rangle \ , \tag{4.1}$$

where currents $J_{a,b} \in \mathcal{S}_n$. The dimension of such matrix is equal to the number of currents in the respective set, i.e., 2^n. To keep things clear, we will be using the following notation: matrices elements will always be written with explicit indices whereas matrices themselves will be indicated by boldface ($\Pi_{ab} \leftrightarrow \mathbf{\Pi}$).

The large N_c limit plus the assumption of confinement guarantees that every current generates only single meson states; the widths go to zero as N_c approaches infinity. Thus, the spectral decomposition of the correlator is given by

$$J_a(t,\vec{x})|0\rangle = \sum_k \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^{3/2}} \ c_{ak} \ \frac{1}{\sqrt{2E_k}} \ e^{i(E_k t - \vec{p}\cdot\vec{x})} \ |k,\vec{p}\rangle \ . \tag{4.2}$$

Here, c_{ak} is the amplitude that the current a creates the particle k.

Using eq. (4.2) we can write the matrix $\mathbf{\Pi}$ (without loss of generality we can take $y=0$)

$$\Pi^{(n)}_{ab}(t,\vec{x}) = \sum_k \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \ C_{ab,k} \ \Delta(t,\vec{x};m_k) \ , \tag{4.3}$$

where $C_{ab,k} = c^*_{ak} c_{bk}$ is the matrix of coefficients and $\Delta(t,\vec{x};m_k)$ is the propagator for a noninteracting scalar of mass m_k . Our matrix can be viewed as the Kallen-Lehmann spectral representation with the spectral function ρ proportional to Dirac delta functions. It is a straightforward consequence of the large N_c [9, 10] limit, planarity of diagrams, and confinement.

We will study only a correlation in time ($\vec{x}=0$) and perform an analytic continuation to an imaginary time ($\tau = it$), thus

$$\Pi^{(n)}_{ab}(\tau) = \sum_k \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \ C_{ab,k} \ \ \Delta(\tau;m_k) \ . \tag{4.4}$$

5 The relation between matrix of correlators and masses of mesons

In this section we derive the eq. (2.4) which is at the heart of the demonstration of a Hagedorn spectrum. We note that a derivation of this was given in ref. [16]. Unfortunately that derivation contained an error. The result, however, is correct and a valid derivation is given here.

Before proceeding, it is useful to recall that it is standard to use current-current correlation functions in lattice QCD to extract the lattice hadronic state in a given channel [8]. The following relations are used in this context:

$$\lim_{\tau\to\infty} -\frac{\mathrm{d}}{\mathrm{d}\tau} \log \langle J(\tau) J(0)\rangle \ = m_0 \ , \tag{5.1}$$

$$-\frac{\mathrm{d}}{\mathrm{d}\tau} \log \langle J(\tau) J(0)\rangle \ > m_0 \ , \tag{5.2}$$

where m_0 is the lowest mass state. The goal here is to generalize the relation in eq. (5.2) to the case of a matrix of correlators $\mathbf{\Pi}^{(n)}$ in large N_c QCD.

From eq. (4.4), the matrix elements are

$$\Pi_{ab}^{(n)}(\tau) = \left\langle J_a^\dagger(\tau) J_b(0) \right\rangle = \sum_k \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\, C_{ab,k}\, \frac{1}{2E_k}\, e^{-E_k\tau} \,, \tag{5.3}$$

where $a, b = 1 \dots 2^n$. We study the following expression and will prove that it is greater than the sum of the lowest 2^n masses with scalar quantum numbers.

$$V^{(n)} \equiv -\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\log \mathbf{\Pi}^{(n)} = \mathrm{Tr}\left(-\dot{\mathbf{\Pi}}^{(n)} {\mathbf{\Pi}^{(n)}}^{-1}\right) \,, \tag{5.4}$$

where the dot stands for the derivative with respect to Euclidean time τ.

First, let us define the following quantity:

$$-\check{\Pi}_{ab}^{(n)}(\tau) \equiv \sum_k \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\, C_{ab,k}\, \frac{1}{2E_k}\, e^{-E_k\tau}\, m_k \,, \tag{5.5}$$

compared to the derivative with respect to τ:

$$-\dot{\Pi}_{ab}^{(n)}(\tau) = \sum_k \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\, C_{ab,k}\, \frac{1}{2E_k}\, e^{-E_k\tau} \sqrt{p^2 + m_k^2} \,, \tag{5.6}$$

One can easily prove that the trace (5.4) with $\check{\mathbf{\Pi}}$ instead of $\dot{\mathbf{\Pi}}$ is always smaller than the trace with the dotted matrix:

$$\begin{aligned} \mathrm{Tr}\left(-\dot{\mathbf{\Pi}}\right)\mathbf{\Pi}^{-1} &\geq \mathrm{Tr}\left(-\check{\mathbf{\Pi}}\right)\mathbf{\Pi}^{-1} \,, \\ \sum_c \langle \psi_c| - \dot{\mathbf{\Pi}} |\psi_c\rangle \lambda_c^{-1} &\geq \sum_c \langle \psi_c| - \check{\mathbf{\Pi}} |\psi_c\rangle \lambda_c^{-1} \,, \end{aligned} \tag{5.7}$$

where $|\psi_c\rangle$ are eigenvectors of matrix $\mathbf{\Pi}$, and λ_c are corresponding eigenvalues. All eigenvalues, λ_c, are positive since $\mathbf{\Pi}$ is a positive definite matrix. The left part of inequality (5.7) represents the average of energy (which includes momenta) in certain states, which is always greater than an average of the analogous masses, that correspond to the term on the right-hand side. The average is taken with positive weights λ_c^{-1} so the inequality holds overall.

Next, we split the matrix $\mathbf{\Pi}$ into two parts, $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}$ couples only to the first 2^n masses and $\mathbf{B}$ to the rest.

$$\begin{aligned} A_{ab}^{(n)}(\tau) &\equiv \sum_{k=1}^{2^n} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\, C_{ab,k}\, \frac{1}{2E_k}\, e^{-E_k\tau} \,, \\ B_{ab}^{(n)}(\tau) &\equiv \sum_{k=2^n+1}^{\infty} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\, C_{ab,k}\, \frac{1}{2E_k}\, e^{-E_k\tau} \,. \end{aligned} \tag{5.8}$$

One can easily see that such splitting can be done for the matrix itself, $\mathbf{\Pi} = \mathbf{A} + \mathbf{B}$, its derivative, $\dot{\mathbf{\Pi}} = \dot{\mathbf{A}} + \dot{\mathbf{B}}$, as well as for the matrix, $\check{\mathbf{\Pi}} = \check{\mathbf{A}} + \check{\mathbf{B}}$.

Our goal is to show that we can relate the $\mathbf{\Pi}$ matrix to the meson masses. We start with the following simple matrix identity (which is valid for every n):

$$\begin{aligned}\mathrm{Tr}\left((-\check{\mathbf{A}}-\check{\mathbf{B}})\,(\mathbf{A}+\mathbf{B})^{-1}\right) &= \mathrm{Tr}\left(\frac{1}{\sqrt{\mathbf{A}+\mathbf{B}}}\sqrt{\mathbf{A}}\left(\frac{1}{\sqrt{\mathbf{A}}}(-\check{\mathbf{A}})\frac{1}{\sqrt{\mathbf{A}}}\right)\sqrt{\mathbf{A}}\frac{1}{\sqrt{\mathbf{A}+\mathbf{B}}}\right)\\ &+ \mathrm{Tr}\left(\frac{1}{\sqrt{\mathbf{A}+\mathbf{B}}}\sqrt{\mathbf{B}}\left(\frac{1}{\sqrt{\mathbf{B}}}(-\check{\mathbf{B}})\frac{1}{\sqrt{\mathbf{B}}}\right)\sqrt{\mathbf{B}}\frac{1}{\sqrt{\mathbf{A}+\mathbf{B}}}\right)\end{aligned} \tag{5.9}$$

Next we introduce the definitions

$$\begin{aligned}\mathbf{X} &\equiv \sqrt{\mathbf{A}}\frac{1}{\sqrt{\mathbf{A}+\mathbf{B}}}\ , & \mathbf{Y} &\equiv \sqrt{\mathbf{B}}\frac{1}{\sqrt{\mathbf{A}+\mathbf{B}}}\ ,\\ \underline{\mathbf{A}} &\equiv \frac{1}{\sqrt{\mathbf{A}}}(-\check{\mathbf{A}})\frac{1}{\sqrt{\mathbf{A}}}\ , & \underline{\mathbf{B}} &\equiv \frac{1}{\sqrt{\mathbf{B}}}(-\check{\mathbf{B}})\frac{1}{\sqrt{\mathbf{B}}}\ .\end{aligned} \tag{5.10}$$

which allows us to recast eq. (5.9) in the following way:

$$\mathrm{Tr}\left((-\check{\mathbf{A}}-\check{\mathbf{B}})\,(\mathbf{A}+\mathbf{B})^{-1}\right) = \mathrm{Tr}\left(\mathbf{X}^\dagger\underline{\mathbf{A}}\mathbf{X}+\mathbf{Y}^\dagger\underline{\mathbf{B}}\mathbf{Y}\right)\ . \tag{5.11}$$

The relation between $\mathbf{X}$ and $\mathbf{Y}$ operators $\mathbf{X}^\dagger\mathbf{X}+\mathbf{Y}^\dagger\mathbf{Y}=1$ guarantees that they can be simultaneously diagonalized and their eigenvalues are related;

$$\mathbf{X}^\dagger\mathbf{X}|\psi_i\rangle = p_i|\psi_i\rangle\ ,\quad \mathbf{Y}^\dagger\mathbf{Y}|\psi_i\rangle = (1-p_i)|\psi_i\rangle\ , \tag{5.12}$$

with $0\le p_i\le 1$. We can see that the states defined as $|\chi_i\rangle \equiv 1/\sqrt{p_i}\,X|\psi_i\rangle$ forms an orthonormal basis of states, as do the states $|\Upsilon_i\rangle \equiv 1/\sqrt{1-p_i}\,Y|\psi_i\rangle$. Thus, we can rewrite the trace into the following form.

$$\begin{aligned}\mathrm{Tr}\left(\mathbf{X}^\dagger\underline{\mathbf{A}}\mathbf{X}+\mathbf{Y}^\dagger\underline{\mathbf{B}}\mathbf{Y}\right) &= \sum_i\langle\psi_i|\mathbf{X}^\dagger\underline{\mathbf{A}}\mathbf{X}+\mathbf{Y}^\dagger\underline{\mathbf{B}}\mathbf{Y}|\psi_i\rangle\\ &= \sum_i\langle\chi_i|\underline{\mathbf{A}}|\chi_i\rangle + (1-p_i)\left(\langle\Upsilon_i|\underline{\mathbf{B}}|\Upsilon_i\rangle - \langle\chi_i|\underline{\mathbf{A}}|\chi_i\rangle\right) \qquad (5.13)\\ &\ge \mathrm{Tr}\underline{\mathbf{A}} = \mathrm{Tr}\frac{-\check{\mathbf{A}}}{\mathbf{A}}\ . \qquad (5.14)\end{aligned}$$

The second term of the right-hand side of eq. (5.13) is always positive since the biggest possible eigenvalue of operator $\underline{\mathbf{A}}$ is less than or equal to m_{2^n} (where m_j is the mass of the j^{th} state) , while the smallest eigenvalue of $\underline{\mathbf{B}}$ is always greater or equal to m_{2^n+1}. This is a straightforward consequence of eq. (5.5), (5.8), and (5.10).

The currents entering operator $\mathbf{A}$ can be reorganized in such a way that the first current couples only to the first meson, second current only to the second meson, etc. It is obvious that this process will lead to the diagonal matrix whose every element is equal to the mass of the n^{th} particle. Consequently

$$\mathrm{Tr}\frac{-\check{\mathbf{A}}^{(n)}}{\mathbf{A}^{(n)}} = \sum_k^{2^n} m_k\ . \tag{5.15}$$

The quantity on the right-hand side is exactly the function $W(m_{2^n})$ defined by eq. (2.1). Consequently, we derived the inequality of (2.4) required for the establishing of a Hagedorn spectrum.

$$V = -\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\log\mathbf{\Pi} \geq \sum_k^{2^n} m_k = W(m_{2^n}) \; . \tag{5.16}$$

6 Matrix of correlators in an asymptotically free regime

In this section, we derive the condition (2.3) stating that the trace of the logarithm of a corellator matrix grows at most linear in n (where n labels the step in the sequence).

For sufficiently small times, asymptotic freedom allows us to treat fields inside the currents as non-interacting, so we can decompose the correlator to a product of single-particle propagator functions. Additionally, the large N_c limit, in which we are working, guarantees that the whole matrix of correlators is diagonal. Doing the trace, we obtain 2^n terms with the same structure. Consequently, we can focus solely on the one current-current correlator and its logarithmic derivative, and investigate how it grows with n.

In the asymptotically free region, the structure of the overall correlator is given simply by the dimensional analysis. Since the time scale τ is the only dimensional parameter left, and our current J has the mass dimension $4n+3$ for 3+1 dimensions, and $12n+3$ for 2+1 dimensions, the right-hand side equals $8n+6$, and $24n+6$, respectively. Thus

$$\Pi_{ab}^{(n)} = \begin{cases} \delta_{ab}\,\mathrm{const}\;\tau^{-8n+6} & \text{for } 3+1\,\text{dimensions} \\ \delta_{ab}\,\mathrm{const}\;\tau^{-24n+6} & \text{for } 2+1\,\text{dimensions} \end{cases} \tag{6.1}$$

and the trace of logarithmic derivative equals

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\log\mathbf{\Pi}^{(n)} = \begin{cases} (2^n)\frac{8n+6}{\tau} & \text{for } 3+1\,\text{dimensions} \\ (2^n)\frac{24n+6}{\tau} & \text{for } 2+1\,\text{dimensions} \end{cases} \tag{6.2}$$

If we assume that the matrix of correlators is effectively at its asymptotic value up to some small corrections for $\tau < \tau_0$ with τ_0 independent of n, we reproduced exactly the inequality condition (2.3) necessary for establishing the Hagedorn spectrum.

From the argument in section 2, the preceding implies a Hagedorn spectrum where the value of Hagedorn temperature corresponding to our sets of currents is

$$T_H \leq \begin{cases} \frac{8\log_2(e)}{\tau_0} & \text{for } 3+1\,\text{dimensions} \\ \frac{24\log_2(e)}{\tau_0} & \text{for } 2+1\,\text{dimensions} \end{cases} \tag{6.3}$$

At this stage, we have shown that a Hagedorn spectrum emerges in the QCD in 3+1 and 2+1 dimensions if a certain assumption is met—namely that correlators are to good approximation at their asymptotically free value for $\tau < \tau_0$ for all n.

7 Perturbative corrections

In the previous section, we neglected all possible interactions between gluons. Such an assumption was justified by the fact that the QCD is in asymptotically free regime for

short times. Provided that this condition is met, we have proved QCD has a Hagedorn spectrum. The critical question is then the circumstances for which there are no large corrections to the asymptotically free result.

The standard way to include the effects of interactions for the correlators at short times is via perturbation theory. However, the region where perturbation theory is valid is certainly limited; as the time increases perturbative corrections grow and ultimately push the system outside the region of validity of perturbation theory. Here we will rely on the standard assumption that perturbation theory accurately describes correlation functions provided that they are small. That is, in the region where pertrubative corrections are small they will dominate over all nonperturbative effects. We note that this is not a rigorous mathematical theorem but it is the basis of standard analysis of QCD correlation functions.

Given this assumption, the critical issue we need to address is how do perturbative corrections scale with n? The goal is to show that at fixed τ perturbative corrections, at any fixed order, to the quantity $2^{-n}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\log\Pi^{(n)}$ scales at most linearly with n for any fixed τ (in order to satisfy the inequality (2.3)). If we can demonstrate this, we have demonstrated a Hagedorn spectrum given the assumptions stated above. To see why, imagine doing the following calculation: start at some fixed but large n and compute $2^{-n}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\log\Pi^{(n)}$ in perturbation theory to some order. Next decrease the value of τ so that perturbative corrections are sufficiently small that the quantity is close to its asymptotically free value, up to corrections which are a small fraction of the total. It is always possible to find a value of τ for which this is true since the system becomes asymptotically free as $\tau \to 0$. By assumption this is the regime in which perturbation theory is trustworthy. Having done this at some fixed value of n, one next increases n keeping τ fixed. Since the perturbative corrections to $2^{-n}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\log\Pi^{(n)}$ have been demonstrated to scale at most as n and since, as seen in eq. (6.2), the leading behavior from the asymptotically free region also scales with n, the fraction size of the correction is independent of n. Since the corrections to the asymptotically free result were small at the original n, they remain small at all n including in the limit of $n \to \infty$. This is sufficient to show a Hagedorn spectrum, given the assumptions stated above, and given the result of section 6.

In the remainder of this section we show that perturbative corrections to $2^{-n}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\log\Pi^{(n)}$ do, in fact, scale with n no faster than linearly and thereby complete the demonstration. We note that an argument that this is the case was presented in ref. [16]. In that work, the results of certain classes of diagrams were presented and shown to be consistent with the needed result. It is was suggested that the structure of the quantity ought to ensure that result continued to hold for all classes of diagram, however, no demonstration of that was given. Here we construct a general argument why the result will hold for all classes of diagram.

In order to get more insight into the correlation functions, we first look at the contribution from non-interacting gluons such as in the Feynman diagram illustrated in figure 2. It will allow us to extract some properties that we will generalize later when we include interactions. In the non-interacting case, the correlator matrix is diagonal due to large N_c

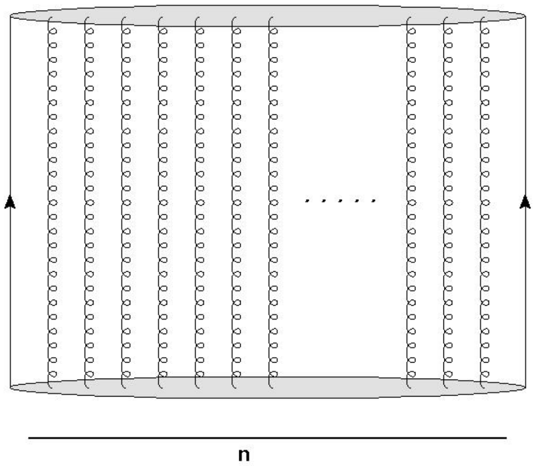

Figure 2. The Feynman diagram for a correlator in the asymptotically free regime. The blobs indicate the currents.

limit. Each propagating gluonic operator O_l inside the current contributes the same way

$$\Pi^1_{ij\,\mathrm{free}}(\tau) = \langle O_i(\tau)\,O_j(0)\rangle_{\mathrm{free}} = \delta_{ij}\,\pi^1_{\mathrm{free}}(\tau)\ , \tag{7.1}$$

where the superscript 1 indicates that this represents the propagation of a single gluonic operator. The δ_{ij} is a result of the choice of operators (3.1), (3.2) which were picked precisely because of this property. The fact that the free correlator is the same for both operators is a result of dimensional analysis which requires them to be proportional to each other; a choice of the constants defining the operators can fix the proportionality constant to unity. We want to emphasize that this object is not gauge invariant, however, it will be only a part of the overall correlator which will be gauge invariant. The small letter π indicates that the quantity is already a function, not a matrix. Such convention will be applied also in the following text. Let us also remind the reader that the matrix elements explicitly indicated indices, whereas the analogous matrices are denoted in boldface.

The correlation function of the whole current consists of n internal lines and is bounded by two quark lines. Thus its matrix elements are given by

$$\Pi^{(n)}_{ab}(\tau) = \delta_{ab}\,\pi^q_{\mathrm{free}}(\tau)\left[\pi^1_{\mathrm{free}}(\tau)\right]^n \pi^q_{\mathrm{free}}(\tau)\ . \tag{7.2}$$

where $\pi^q_{\mathrm{free}}(\tau)$ is the free quark propagator traced over Dirac indices. Recall that the indexes a, b goes from 1 to 2^n. The quantity of interest is the derivative of trace of the logarithm,

$$2^{-n}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\,\log\boldsymbol{\Pi}^{(n)} = 2\frac{\mathrm{d}}{\mathrm{d}\tau}\log\pi^q_{\mathrm{free}} + n\frac{\mathrm{d}}{\mathrm{d}\tau}\log\pi^1_{\mathrm{free}}\ . \tag{7.3}$$

The first term is independent of n (and comes for 2 quarks on the boundaries), while the second one is linearly proportional to n. This result is in agreement with the results obtained in the previous section.

Diagrams with interactions are more complicated. Let us work in a case, when all interaction up to order α^l are included. First, we restrict our attention to the case where

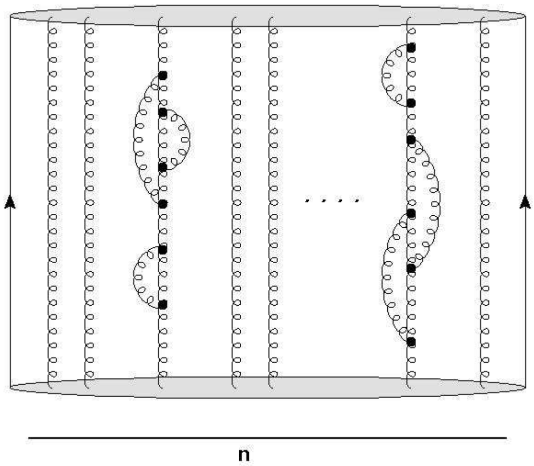

Figure 3. An example of a Feynman diagram where interactions do not couple distinct gluon lines connected to the sources.

all of the interactions act on single gluons that are connected to the sources (see figure 3). While more than one of these gluons may be involved, there are no interactions which couple distinct gluons coupled to the source in this class. Effectively, these interactions lead to the modification of a free propagator similar to the contribution of self-energy correction.

$$\pi^1_{\rm free}(\tau) \to \pi^1_{\rm free}(\tau)\ (1 + c\,(\tau))\ , \tag{7.4}$$

where $(1 + c(\tau))$ is a pertrubative correction. Note that c depends on the order to which we work in perturbation theory, but it is well defined at any given order. Such correction can appear on any internal gluonic line, so that if these were the only types of diagrams contributing we could write the total correlator as

$$\Pi^{(n)}_{ab}(\tau) = \delta_{ab}\ \pi^q_{\rm free}(\tau)\left[\pi^1_{\rm free}(\tau)\,(1 + c(\tau))\right]^n \pi^q_{\rm free}(\tau)\ . \tag{7.5}$$

This structure actually contains more information (higher order in α) than necessary, but certainly contains all combinations that are required to order α^l. The trace of the logarithm now reads

$$2^{-n}\frac{\rm d}{{\rm d}\tau}{\rm Tr}\ \log\mathbf{\Pi}^{(n)} = 2\frac{\rm d}{{\rm d}\tau}\log\pi^q_{\rm free} + n\frac{\rm d}{{\rm d}\tau}\log\pi^1_{\rm free} + n\frac{\rm d}{{\rm d}\tau}\log(1 + c)\ . \tag{7.6}$$

Although we have yet to calculate c, it is clearly independent of n. So the total expression grows, again, at most linearly with n. Of course, this result is wrong—the class of diagrams we considered was chosen artificially and does not correspond to the full perturbative result at any order in α. However, it does illustrate how the logarithmic structure combined with factorizing point-to-point correlators for the individual lines yields total perturbative corrections which grow at most with n. Our main task is to show that a structure enriched with inter-gluonic interactions obeys the same at most linear n dependant growth.

Up to now, we have ignored possible interactions between gluons. In order to deal with them in a simple way, we artificially divide the total number of internal gluon lines n into

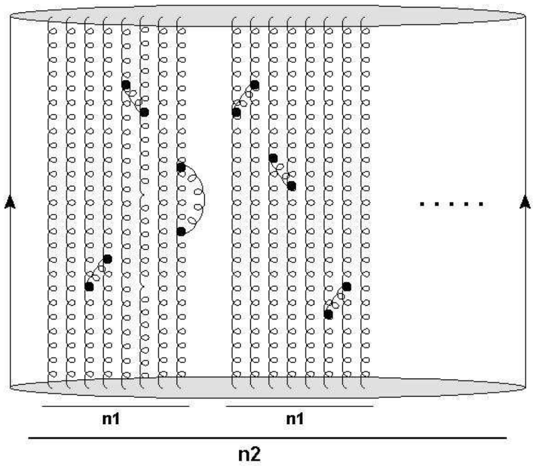

Figure 4. Division of gluon lines into n_2 clusters each containing n_1 lines.

n_2 clusters each containing n_1 gluon lines, that is $n = n_1 n_2$. We will impose the conditions $n_1 \gg 1$, $n_2 \gg 1$; and $n_1 \gg$ the order in perturbation theory to which we are working. For the beginning, let us assume that all interactions occur within clusters, as is illustrated in the figure 4, i.e., there is no internal line between clusters or between these clusters with quark lines bounding them. Obviously, in doing this we neglect a certain class of diagrams for now. However, we will subsequently show that correction due to their inclusion does not affect the leading n behavior. The reason for this is that for a fixed order of perturbation theory only a very small fraction of all Feynman diagrams will connect different clusters and this fraction is small enough to alter leading behavior.

The propagation within one cluster is given by

$$\mathbf{\Pi}^C(\tau) = \left[\pi^1_{\text{free}}(\tau)\right]^{n_1} \left[\mathbf{1} + \mathbf{C}'(\tau, n_1)\right] \ , \tag{7.7}$$

where $[\mathbf{1} + \mathbf{C}'(\tau, n_1)]$ represents the effect of interactions within one cluster. Note that $\mathbf{\Pi}^C$ is a matrix of the dimension $2^{n_1} \times 2^{n_1}$. The $\mathbf{C}'(\tau, n_1)$ depends on the order to which we work in perturbation theory, and, obviously, it depends on the size of the cluster, the number of internal lines n_1.

Now, we need to define a mapping from the “cluster”space, which has the dimension 2^{n_1} matrix to the “overall correlator” space with the dimension 2^n, $\mathbf{D}(\mathbf{M})$. More precisely, we actually need n_2 different mappings $\mathbf{D}^{(k)}(\mathbf{M})$. each corresponding to a different cluster.

Our mapping will be a $2^n \times 2^n$ matrix $D^{(k)}_{ab}$. Note that indexes a and b can be represented by a sequence of n numbers, $a = (a_1 a_2 \dots a_n)$, $b = (b_1 b_2 \dots b_n)$, where $a_i, b_i = 0, 1$. It is a straightforward consequence of our original definition of currents (3.3), each of them being constructed from n building block operators (translated into n internal gluonic lines in each diagram). Recall that we divided n lines into n_2 clusters of n_1 elements . Moreover since we neglect interactions *between* clusters, we want to treat them independently. Thus it is useful to define a mapping $\mathbf{D}^{(k)}$ to work in such a way that the first one, $\mathbf{D}^{(1)}$, affects only first n_1 subindices within indices a and b, the mapping corresponding to the second

one, $\mathbf{D}^{(2)}$, affects pieces n_1+1 to $2n_1$, etc. Analogously, all n_2 mappings corresponding to all possible n_2 clusters are defined. Specifically,

$$D^{(1)}_{(a_1a_2\ldots a_n)(b_1b_2\ldots b_n)}(\mathbf{M}) = M_{(a_1\ldots a_{n_1})(b_1\ldots b_{n_1})} \prod_{l=n_1+1}^{n} \delta_{a_lb_l} \ ,$$
$$D^{(2)}_{(a_1a_2\ldots a_n)(b_1b_2\ldots b_n)}(\mathbf{M}) = M_{(a_{n_1+1}\ldots a_{2n_1})(b_{n_1+1}\ldots b_{2n_1})} \prod_{l=1}^{n_1} \delta_{a_lb_l} \prod_{l'=2n_1+1}^{n} \delta_{a_{l'}b_{l'}} \ ,$$

and the general form

$$D^{(k)}_{(a_1a_2\ldots a_n)(b_1b_2\ldots b_n)}(\mathbf{M}) = M_{(a_{(k-1)n_1+1}\ldots a_{kn_1})(b_{(k-1)n_1+1}\ldots b_{kn_1})} \prod_{l=1}^{(k-1)n_1} \delta_{a_lb_l} \prod_{l'=kn_1+1}^{n} \delta_{a_{l'}b_{l'}} \ , \tag{7.8}$$

where $\mathbf{M}$ is an $n_1 \times n_1$ matrix. Note that such matrices are a straightforward generalization of the previous case where the equivalent of $\mathbf{\Pi^C}$ was just number (matrix 1×1) and the matrix $\mathbf{D}$ was diagonal. It is worth mentioning that the 2^n dimensional overall space can be factorized as a product of n_2 subspaces with dimensions 2^{n_1} with each subspace corresponding to one particular cluster. So, the vectors have the form

$$|V\rangle = |v^{(1)}\rangle \otimes |v^{(2)}\rangle \otimes \cdots \otimes |v^{(n_2)}\rangle \ , \tag{7.9}$$

and the spirit of the mapping $\mathbf{D}^{(k)}$ is

$$\mathbf{D}^{(k)}(\mathbf{M}) = \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} \otimes \cdots \otimes \mathbf{M}^{(k)} \otimes \cdots \otimes \mathbf{1}^{(n_2)} \ . \tag{7.10}$$

From the construction of $\mathbf{D}^{(k)}(\mathbf{M})$ it is easy to show that $\mathrm{Tr}\,\log \mathbf{D}^{(k)}(\mathbf{M}) = \mathrm{Tr}\,\log \mathbf{M}$.

Using this notation and imposing the condition that we neglect all diagrams connecting clusters, the overall correlator contains a product of n_2 matrices $\mathbf{D}^{(k)}$ corresponding to the respective clusters

$$\mathbf{\Pi}^{(n)} = \pi^q_{\text{free}}(\tau)\ \ \mathbf{D}^{(1)}(\mathbf{\Pi}^C)\ \ \times\ \ \mathbf{D}^{(2)}(\mathbf{\Pi}^C)\ \ \ldots\ \ \mathbf{D}^{(n_2)}(\mathbf{\Pi}^C)\ \ \pi^q_{\text{free}}(\tau)\ . \tag{7.11}$$

Using the general property that $\mathrm{Tr}\,\log(\mathbf{AB}) = \mathrm{Tr}\,\log \mathbf{A} + \mathrm{Tr}\,\log \mathbf{B}$ and the property that $\mathrm{Tr}\,\log \mathbf{D}^{(k)}(\mathbf{M}) = \mathrm{Tr}\,\log \mathbf{M}$, it is straightfoward to show that

$$2^{-n}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\,\log \mathbf{\Pi}^{(n)} = 2\frac{\mathrm{d}}{\mathrm{d}\tau}\log \pi^q_{\text{free}} + n_1n_2\frac{\mathrm{d}}{\mathrm{d}\tau}\log \pi^1_{\text{free}} + n_2\frac{\mathrm{d}}{\mathrm{d}\tau}\log(1+c'(n_1)) \tag{7.12}$$

where $c'(n_1) \equiv \exp\left(\mathrm{Tr}\,\log\left(\mathbf{1}+\mathbf{C}'\right)\right) - 1$.

The first two terms were already discussed after eq. (7.3). The second term becomes obvious once one recalls that $n_1n_2 = n$. The third term requires more care. We can denote the expression $\frac{\mathrm{d}}{\mathrm{d}\tau}\log(1+c'(n_1))$ as $f(n_1)$, i.e., some at present unknown function of n_1. However, our choice of clusters is completely arbitrary. Provided our assumption that diagrams connecting clusters does not affect the leading behavior is correct, we can switch n_1 and n_2 and the result must remain unchanged. This requires $n_2f(n_1) = n_1f(n_2)$ and

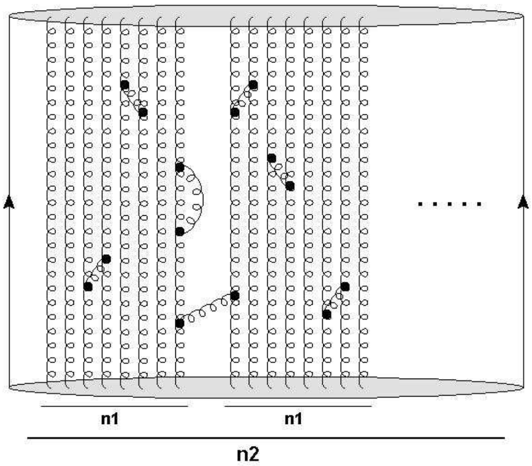

Figure 5. An example of a Feynman diagram with interaction between two clusters.

consequently $f(m)$ must be a linear function of m. Thus, the third term on the right-hand side of eq. (7.12) is also proportional to $n = n_1 n_2$, just as in eq. (7.6).

To complete the demonstration we need to show that the inclusion of diagrams with interactions connecting individual clusters does not affect the leading scaling. An example of such diagram is in figure 5. These effects can be accounted for by including a matrix $(\mathbf{1} + \mathbf{C}'')$ between the matrices corresponding to clusters. Additionally, one should include the interaction between the quarks on the boundary and the neighboring clusters of gluons $(\mathbf{1} + \mathbf{C}^q)$. The total correlator matrix reads

$$\begin{aligned}\mathbf{\Pi}^{(n)}(\tau) = \pi^q_{\text{free}}(\tau)\ (\mathbf{1}+\mathbf{C}^q)\ \times\ \mathbf{D}^{(1)}(\mathbf{\Pi}^C)\ \times\ (\mathbf{1}+\mathbf{C}'')\ \times\ \mathbf{D}^{(2)}(\mathbf{\Pi}^C)\ \ldots \\ (\mathbf{1}+\mathbf{C}'')\ \times\ \mathbf{D}^{(n_2)}(\mathbf{\Pi}^C)\ \times\ \mathbf{D}^{(1)}(\mathbf{\Pi}^C)\ (\mathbf{1}+\mathbf{C}^q)\ \pi^q_{\text{free}}(\tau)\,. \end{aligned} \tag{7.13}$$

The key point here is that the matrices $\mathbf{C}''$ and $\mathbf{C}^q$ must be independent of n_1. The reason for this is that by construction, n_1 is much larger than the order in perturbation theory to which we are working. At large N_c only planar diagrams contribute. Thus if we are working at order α_s^l, a diagram connecting two clusters can at most go $l-1$ gluon lines into the cluster. Since this is less than n_1 it cannot go across the cluster. Thus, the dynamics in $\mathbf{C}''$ does not know how large the cluster is and must be independent of n_1. It is clearly independent of n_2 either.

Using an analogous argument as earlier, the trace of the logarithm reads

$$\begin{aligned}2^{-n}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Tr}\,\log\mathbf{\Pi}^{(n)} = 2\frac{\mathrm{d}}{\mathrm{d}\tau}\log\left(\pi^q_{\text{free}}(1+c^q)\right) + n_1 n_2\frac{\mathrm{d}}{\mathrm{d}\tau}\log\pi^1_{\text{free}} \\ +n_2\frac{\mathrm{d}}{\mathrm{d}\tau}\log(1+c'(n_1)) + (n_2-1)\frac{\mathrm{d}}{\mathrm{d}\tau}\log(1+c'')\end{aligned} \tag{7.14}$$

where c^q and c'' are defined analogously to c'. The first term does not scale with n; the second term is directly proportional to n. Defining $f(n_1) \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\log(1 + c'(n_1))$ and $g \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\log(1+c'')$, the last two terms can be written as $n_2(f(n_1)+g)-g$. Again exploiting

the fact that we can switch n_1 and n_2 without affecting the result (provided the order in perturbation theory is less than both n_1 and n_2) we obtain the consistency condition that

$$n_2(f(n_1) + g) = n_1(f(n_2) + g) \tag{7.15}$$

which yields $f(m) = bm - g$ where b is a constant. Thus, the effect of including the interactions between the clusters simply fixes the subleading behavior in f. Taking the last two terms together and exploiting the fact that $n = n_1 n_2$ we have $bn - g$ which grows linearly in n. Consequently, the right-hand side of eq. (7.14) grows at most linearly with n, and the inequality (2.3) is satisfied if the interactions are included via perturbation theory at any fixed order.

With this we have completed our demonstration that a Hagedorn spectrum arises in large N_c QCD in both 2+1 and 3+1 dimensions. This demonstration depends on one critical assumption: that perturbation theory accurately describes the trace of the logarithm of a matrix of point-to-point correlation functions in the regime where the perturbative corrections to the asymptotically free value are small.

Acknowledgments

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